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No 4

SOME PROBLEMS IN LOGICAL MODEL-THEORY

BY

LARS SVENONIUS

Uppsala

CWK GLEERUP
LUND

EJNAR MUNKSGAARD
COPENHAGEN

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AKADEMISK AVHANDLING

som med tillstånd av vittberömda humanistiska fakultetens i Uppsala historisk-filosofiska sektion för vinnande av filosofie doktorsgrad till offentlig granskning framställes å lärosal V

FREDAGEN 20 MAJ 1960 KL. 10 F.M.

AV

LARS SVENONIUS
Fil. lic., Norrl.

UPPSALA 1960
Almqvist & Wiksells Boktryckeri AB

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PRINTED IN SWEDEN BY

Almqvist & Wiksells

BOKTRYCKERI AKTIEBOLAG

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ACKNOWLEDGEMENTS

Among those who have helped me in different ways in producing this essay I wish to mention my teacher, professor Konrad Marc-Wogau, my friend Lars-Erik Sanner, and my wife Elaine.

I also wish to thank Dr. William Craig, who kindly read through my manuscript and made a number of suggestions. Unfortunately, there was not time to incorporate them in the printed version.

1. INTRODUCTION

The content of this essay can be described as follows: We present a theory of some algebraic languages, called "elementary languages" (3.3. D), and we show this theory to be equivalent in a certain sense to the theory of languages in first-order predicate calculus with identity. (4. 15. T). This algebraic theory is similar to those described by Halmos (4) and by Tarski-Thompson (18), but there are some non-trivial differences. We also study a much more general class of languages, called $\mathfrak{L}$-languages. We define for this general class a number of concepts, generalizing concepts currently applied to elementary languages (languages in first-order calculus). We prove some general model-theoretic theorems about $\mathfrak{L}$-systems, and restate some known theorems about first-order languages as theorems about elementary systems.

It may be noted that this essay also summarizes the main results of the author's papers (13), (14) and (15).

2. RELATIONS, REGULAR OPERATIONS

2.1. D.[1] *Relation of degree n with universe U; d(R), U(R):* If U is a class. and U^n (as usual) denotes the class of n-termed sequences of elements in U, then by a *relation of degree n with universe U* we mean a function R from U^n to a two-element class $\{0, 1\}$. (The elements 1 and 0 may be regarded as representing truth and falsity, respectively.) If R is a relation, the degree of R is denoted $d(R)$ and the universe of R is denoted $U(R)$. In the case that $d(R)$ is 0, the function R is a constant and is either 0 or 1.

2.2. D. *Isomorphic relations:* If R is an n-degree relation with universe U, and π is a one-one mapping from U onto some class U', we denote by $\pi(R)$ the relation R' with universe U' such that $R'(a_1, \ldots, a_n) = R(\pi^{-1}(a_1), \ldots, \pi^{-1}(a_n))$. We call π an *isomorphism* mapping of R to R', and we say that R and R' are *isomorphic relations.*

2.3. D. *Automorphism; automorphism group:* If R is a relation with universe U, π a one-one mapping of U onto itself such that $\pi(R) = R$, we call π an *automorphism of R*; the class (group) of all such mappings we call the *automorphism-group of R.*

2.4. D. *Regular operations on relations:* We consider operations which for some k, $(k = 1, 2, \ldots)$ when applied to some k-tuples of relations $(R_1, \ldots, R_k)$ with a common universe U, give another relation with the same universe. We limit our attention to "regular" operations, defined as follows: A k-ary

[1] In each section, we have numbered consecutively those paragraphs which contain definitions or theorems. A '*D*' after the number means "definition(s)", and a '*T*' means "theorem".

operation P (on relations to relations) is said to be *regular*, if it satisfies the following three conditions:

(a) Whenever $P(R_1, \ldots, R_k)$ is defined, the relations $R_1, \ldots, R_k$, and $P(R_1, \ldots, R_k)$, are relations with a common universe.

(b) Whenever P is defined for a k-tuple $(R_1, \ldots, R_k)$, and d is the degree of $P(R_1, \ldots, R_k)$ and $(R'_1, \ldots, R'_k)$ is a k-tuple of the same type as $(R_1, \ldots, R_k)$—i.e. such that $d(R_1) = d(R'_1)$, $\ldots$, $d(R_k) = d(R'_k)$—then P is defined for $(R'_1, \ldots, R'_k)$ and d is the degree of $P(R'_1, \ldots, R'_k)$.

(c) If $P(R_1, \ldots, R_k) = R_0$, where U is the universe of the relations $R_1, \ldots, R_k$, R_0, and if π is a one-one mapping of U onto some class U', then $P(\pi(R_1), \ldots, \pi(R_k)) = \pi(R_0)$.

2.5. D. *Type-sequence of a (regular) operation:* If for some $R_1, \ldots, R_k, R_0$, $R_0 = P(R_1, \ldots, R_k$, then the sequense $(d(R_1), \ldots, d(R_k), d(R_0))$ will be called a *type-sequence* of P.

2.6. D. *The class Q_0 of fundamental elementary operations:* The operations defined under (1)–(6) below will be called the *fundamental elementary operations.* The class of these operations will be designated Q_0.—It can easily be verified that they are all regular operations.

(1) When σ is a permutation of the class $\{1, \ldots, n\}$, we let the operation Per_σ (applicable to relations of degree n) be defined by the condition: When R is a relation of degree n, $Per_\sigma(R)$ is the relation R', of degree n and with the same universe as R, such that $R'(a_1, \ldots, a_n) = R(a_{\sigma^{-1}(1)}, \ldots, a_{\sigma^{-1}(n)})$. We will call the operations in this class "place-permutations".

(2) We define the operation N (negation) by the condition: When R is a relation, of arbitrary degree n, $N(R)$ is the relation R' of degree n (with the same universe as R) such that $R'(a_1, \ldots, a_n) = 1$ iff $R(a_1, \ldots, a_n) = 0$.

(3) When k is any integer $\geqslant 1$, we let the operation E_k (existential quantifier), applicable to all relations of degree $\geqslant k$, be defined by the condition: When R is a relation of degree $n \geqslant k$, $E_k(R)$ is the relation R' (with the same universe as R), of degree $n-1$, such that $R'(a_1, \ldots, a_{n-1}) = 1$ iff $R(b_1, \ldots, b_n) = 1$

for some sequence $(b_1, \ldots, b_n)$ such that $b_i = a_i$ for $i < k$ and $b_i = a_{i-1}$ for $i > k$, (and b_k is arbitrary). In the case that $k = d(R) = 1$, $E_k(R)$ is the constant 1, or the constant 0, (regarded as functions of degree 0), depending on whether there is or is not an element a in $U(R)$ such that $R(a) = 1$.

(4) If $r, s \geqslant 1$, and $r \neq s$, we let $I_{r,s}$ ("identity") be the operation (applicable to relations of degree $n \geqslant \max(r, s)$) defined by the condition: If R is a relation of degree $n \geqslant \max(r, s)$, then $I_{r,s}(R)$ is the relation R', of degree n and with the same universe as R, such that $R'(a_1, \ldots, a_n) = 1$ iff $R(a_1, \ldots, a_n) = 1$ and $a_r = a_s$.

(5) If $r, s \geqslant 1$, and $r \neq s$, we let $Id_{r,s}$ ("merging") be the operation (applicable to relations of degree $n \geqslant \max(r, s)$) defined by the condition: If R is a relation of degree $n \geqslant \max(r, s)$, then $Id_{r,s}(R)$ is the relation R', of degree $n-1$ and with the same universe as R, such that $R'(a_1, \ldots, a_{n-1}) = R(b_1, \ldots, b_n)$, where $(b_1, \ldots, b_n)$ is the sequence such that $b_i = a_i$ for $i < s$, $b_i = a_{i-1}$ for $i > s$, and $b_s = a_r$.

(6) Finally, the binary operation *Comp* ("composition"), applicable to any pair (R_1, R_2) of relations with a common universe, is defined by the condition: If R_1, R_2 are relations of degrees n and m respectively, and with universe U, then *Comp* (R_1, R_2) is the relation R' of degree $n+m$ and with universe U, such that $R'(a_1, \ldots, a_{n+m}) = 1$ iff $R_1(a_1, \ldots, a_n) = 1$ and $R_2(a_{n+1}, \ldots, a_{n+m}) = 1$.—When $n = d(R_1) = 0$, and $d(R_2) = d$, *Comp* (R_1, R_2) is equal to R_2 if R_1 is the constant 1; and if R_1 is the constant 0, *Comp* (R_1, R_2) is the relation R', of degree d, such that $R'(a_1, \ldots, a_d) = 0$ for all $a_1, \ldots, a_d$ in U.

Note 1. It may be observed that these operations are not independent. For instance, $Id_{r,s}(R)$ is always the same relation as $E_s I_{r,s}(R)$.

Note 2. The operations in Q_0 just defined are all, except for the permutation operations of group 1, applicable to relations of different degrees n. It might be theoretically more satisfactory to say that we have really different operations when the n's are different. We would then say, for instance, that

there are different operations E_2^2, E_2^3, E_2^4 ..., which are all defined as E_2 is in (3), except that E_2^2 is applicable only to relations of degree 2, E_2^3 is applicable only to relations of degree 3, etc.

Note 3. About the concept "regular": Condition (c) in the definition of "regular" may be explained as stating that regular operations take account only of structural properties of the relations involved, and not of the specific elements entering into the relations. It seems intuitively clear, that all operations dealt with in logic are of this kind. This is proved for a class of logical operations in Lindenbaum-Tarski (6).

3. $\mathfrak{L}$-LANGUAGES

We will consider a very general class of algebraic languages, called "$\mathfrak{L}$-languages", which will be interpreted as languages "about" relations. The simplest (although perhaps not the most natural) way to give a general description of these is to regard them as certain subclasses of a very general class of symbols. We will first describe this universal symbol-class, denoted $\mathfrak{L}$.

3.1. D. *Variables, operation-symbols, terms in* $\mathfrak{L}$*;* $d(t)$*:*
(a) (Variables.) For every integer $n \geqslant 0$, $\mathfrak{L}$ is assumed to contain a denumerable number of distinct symbols $r_1^n, r_2^n, \ldots$, called *variables of degree n* in $\mathfrak{L}$.

(b) (Operation-symbols.) We assume that $\mathfrak{L}$ contains a distinct operation-symbol for every regular operation (on relations to relations),—and no other operation-symbols. (We assume that some symbol is available for any regular operation we might want to represent.) If p is an operation-symbol, the operation which p represents will be denoted P_p.

(c) (Terms.) Every variable of degree n (in $\mathfrak{L}$) is also called a *term of degree n* (in $\mathfrak{L}$), for every n. If $t_1, \ldots, t_k$ are terms of degrees $d_1, \ldots, d_k$, respectively, and p is an operation symbol such that $(d_1, \ldots, d_k, d_0)$ is a type-sequence of P_p, then $p(t_1, \ldots, t_k)$ is a term of degree d_0. (This number d_0 is uniquely determined by p and the degrees of $t_1, \ldots, t_k$ according to condition (b), 2.4. D).—If t is a term in $\mathfrak{L}$, we denote by $d(t)$ the degree of t.

3.2. D. *The class* Q_0 *of elementary operation-symbols; elementary terms:* In particular, $\mathfrak{L}$ contains symbols for the fundamental elementary operations (the class Q_0). We will designate these symbols in $\mathfrak{L}$ by symbols in boldface italics similar

to the symbols used above to refer to the operations themselves, as follows: For an arbitrary permutation σ of the class $\{1, \ldots, n\}$, for any n, and for arbitrary integers r, s, $k \geqslant 1$, we denote by $\boldsymbol{Per}_\sigma$, $\boldsymbol{N}$, $\boldsymbol{E}_k$, $\boldsymbol{I}_{r,s}$, $\boldsymbol{Id}_{r,s}$ and $\boldsymbol{Comp}$, the operation symbols of $\mathfrak{L}$ which represent, respectively, the operation Per_σ, N, E_k, $I_{r,s}$ $Id_{r,s}$, and $Comp$.—A term in $\mathfrak{L}$, which only contains elementary operation-symbols, i.e. only operation-symbols in $\boldsymbol{Q}_0$, may be called an *elementary term.*

Examples of elementary terms: $\boldsymbol{E}_2(r_1^2)$ is a term of degree 1; $\boldsymbol{Comp}\ (r_2^0, \boldsymbol{E}_2(r_1^2))$ is a term of degree 1; $\boldsymbol{Per}_\sigma\ (\boldsymbol{Comp}\,(r_1^2, \boldsymbol{N}(r_1^2)))$ is a term of degree 4, if σ is a name of the permutation $\left(\begin{smallmatrix}1&2&3&4\\2&3&4&1\end{smallmatrix}\right)$.

3.3. D. *Generated terms, Gen (C, Q); language in $\mathfrak{L}$, sum of languages, elementary languages, V(L), Q(L), $L_0(V)$:* If C is a class of terms in $\mathfrak{L}$, and Q a class of operation symbols, we may define "the class of terms generated from C by Q", symbolized $Gen\ (C, Q)$, to be the smallest class C' which contains C, and is such that whenever $p \in Q$, $t_1, \ldots, t_k \in C'$, and $p(t_1, \ldots, t_k)$ is a term in $\mathfrak{L}$, then $p(t_1, \ldots, t_k) \in C'$. A class $\mathfrak{L}$ of terms which can be represented as a class $Gen\ (V, Q)$, where V is a class of variables in $\mathfrak{L}$ and Q a class of operation-symbols in $\mathfrak{L}$, will be called a *language in* $\mathfrak{L}$, or an *$\mathfrak{L}$-language* (or, when no confusion is likely, simply a *language*). When L is a language, the class of variables in L will be denoted $V(L)$, and the class of operation-symbols occurring in L will be denoted $Q(L)$. If $\{L_1, L_2, \ldots\}$ is a class of languages, the *sum* of these languages is the language L such that $Q(L)$ is the union $Q(L_1) \cup Q(L_2) \cup \ldots$, and such that $V(L)$ is the union $V(L_1) \cup V(L_2) \ldots$. A language L will be called an *elementary language,* if $Q(L) = \boldsymbol{Q}_0$, and $V(L)$ contains at least one variable of degree $\geqslant 1$. (The last clause is introduced to avoid some trivial cases.) The elementary language L with $V(L) = V$ will be denoted $L_0(V)$.

3.4. D. *Equation; equation in L:* An *equation* (in $\mathfrak{L}$) is an expression of the form $t_1 = t_2$, where t_1 and t_2 are terms in

$\mathfrak{L}$, and both are of the same degree. When L is a language, an *equation in* L is an equation $t_1 = t_2$ where t_1 and t_2 are terms in L.

3.5. D. *Valuation; consistent valuation:* If C is any class of terms, any mapping M from C onto some class of relations with a common universe U, such that, for every n, terms of degree n are mapped on relations of degree n, is called a *valuation of* C *with universe* U. If C is a language L, and for every term $p(t_1, \ldots, t_k)$ in L, $M(p(t_1, \ldots, t_k)) = P_p(M(t_1), \ldots, M(t_k))$, M is said to be a *consistent valuation in* L. If C is an arbitrary class of terms, and M is a valuation of C, M is called a *consistent valution of* C if it can be extended to a consistent valuation of L, where L is the smallest language which includes C. (There is always such a smallest language L.)

3.6. T. If V is a class of variables, it is easy to see that every valuation of V is consistent.—If M is a consistent valuation of C, and C' is a class of terms which includes C, then there is some consistent valuation M' of C' which is an extension of M.

3.7. D. *Induced valuation,* $T_{(C, M)}$, $T(M, t)$*:* If M is any consistent valuation of a set C, and Q is a set of operation-symbols, there is a uniquely determined consistent extension M' of M to the set $C' = Gen(C, Q)$. We will call this "the valuation of C' *induced* by M", and denote it $T_{(C', M)}$. The value of this function for any term t in C', or $T_{(C', M)}(t)$, will also be written $T(M, t)$. (It is not dependent on the particular C' chosen).

3.8. D. *"Hold for", "true for", defined for equations and sets of equations; consequence:* If t_1 and t_2 are any terms, and M is a consistent valuation of a class C, the equation $t_1 = t_2$ is said to *hold for* M, or be *true for* M, if $M'(t_1) = M'(t_2)$ for every consistent extension M' of M to the class $C \cup \{t_1, t_2\}$.

If $\mathcal{E}$ is a class of equations, and M a consistent valuation of a class C, $\mathcal{E}$ is said to *hold for* M, if every member of $\mathcal{E}$ holds for M. If E is an equation and $\mathcal{E}$ a class of equations, then E is said to be a *consequence* of $\mathcal{E}$ if E holds for M whenever $\mathcal{E}$ holds for M.

3.9. D. *System in $\mathfrak{L}$; term syntactically defined from term-set; generating set of terms in a system:* If L is a language and $\mathcal{E}$ a class of equations in L, the ordered pair $(L, \mathcal{E})$ will be called a *system*, (*system in* $\mathfrak{L}$). If $(L, \mathcal{E})$ is a system, $t \in L$, $C \subseteq L$, and $Q = Q(L)$, we will say that t is *syntactically defined* from C in $(L, \mathcal{E})$ if there is some t' in *Gen* (C, Q) such that the equation $t = t'$ is a consequence of $\mathcal{E}$. (Intuitively: it is possible to express, in terms of C and the operation-symbols of L, a term which is equivalent to t under the conditions $\mathcal{E}$.) If every term in L is syntactically defined from C in $(L, \mathcal{E})$, then C is called a *generating set of* $(L, \mathcal{E})$. In particular, of course, the set $V(L)$ of variables in L is a generating set of any system $(L, \mathcal{E})$.

3.10. D. *Model of a system:* If $(L, \mathcal{E})$ is a system in $\mathfrak{L}$, by a *model* of $(L, \mathcal{E})$ we will mean a consistent valuation M of $V(L)$ for which the equation-set $\mathcal{E}$ holds.

Evidently, a model M of $(L, \mathcal{E})$ determines a valuation $T_{(L, M)}$ of the language L, for which $\mathcal{E}$ holds. We might alternatively have defined the model as this valuation of L.

3.11. D. *Consistent system, complete system:* A system $(L, \mathcal{E})$ is called *consistent*, if there exists a model of $(L, \mathcal{E})$, otherwise it is said to be *inconsistent*. A system $(L, \mathcal{E})$ is said to be *complete*, if for every equation E of L which is not a consequence of $\mathcal{E}$, the system $(L, \mathcal{E} \cup \{E\})$ is inconsistent.

3.12. D. *Sequences which satisfy a term:* Let M be a valuation of V (a set of variables) with universe U. Let L be a language with $V(L) = V$, let t be a term in L of degree n, and $(a_1, \ldots, a_n)$ a sequence of elements in U. We will say

that the sequence $(a_1, \ldots, a_n)$ *satisfies* t in M, if, when $R = T(M, t)$, $R(a_1, \ldots, a_n) = 1$.

3.13. D. *Enlargement of a valuation:* Let M and M' be two valuations of some term-class C, with universes U and U', respectively. We will say that M' is an *enlargement* of M, if U is a subset of U', and if for every term t in C (of arbitrary degree n) when $R = M(t)$ and $R' = M'(t)$, $R(a_1, \ldots, a_n) = R'(a_1, \ldots, a_n)$ whenever $a_1, \ldots, a_n \in U$.

Particularly important is the case, when L is a language and the valuation of L induced by one valuation is an enlargement of the valuation of L induced by another valuation. This case is taken care of in the following definition.

3.14. D. *L-enlargement:* Let L be a language with $V(L) = V$, and let M and M' be two valuations of V. We will call M' an *L-enlargement* of M if the valuation $T_{(L, M')}$ of L induced by M' is an enlargement of the valuation $T_{(L, M)}$ of L induced by M.

(In the case of elementary languages, this concept corresponds to the concept of "arithmetic extension" or "elementary extension" of valuations for languages in predicate calculus.[1] This will be seen from the general correspondence between $PC_1 =$ and the elementary languages demonstrated in the next section.)

[1] See Tarski-Vaught (19).

4. ELEMENTARY LANGUAGES AND LANGUAGES IN FIRST-ORDER PREDICATE CALCULUS WITH IDENTITY ($PC_1=$)

4.1. D. *Individual variables, relation symbols, formulas, sentences in* $PC_1=$*:* We will consider a version of $PC_1=$ with equality, conjunction, negation, and existential quantifier as the logical primitives. We will assume that there are denumerably many *individual variables* x_1, x_2, ..., and for each integer $n \geqslant 0$, a denumerable number of *relation symbols of degree n:* R_1^n, R_2^n, ... We may assume as familiar the definitions of *formula* (in $PC_1=$), and *sentence* (in $PC_1=$).

4.2. D. *Language in* $PC_1=$*, primitive symbol; system in* $PC_1=$*:* If V is any class of relation symbols, the class L of formulas which contain only relation symbols from the class V will be called a *language in* $PC_1=$, and will be denoted $L(V)$; the class V will be called the *class of primitive symbols of* L, and will be denoted $V(L)$. By a *system in* $PC_1=$ we will mean a pair $(L, \sum)$, where L is a language and $\sum$ a class of sentences in L.

4.3. D. *Abstract in L; primitive abstract:* If Φ is a formula in a language L, and v_1, ..., v_n[1] are the distinct free individual variables of Φ, the expression $\hat{v}_1 \ldots \hat{v}_n \Phi$ is called an *abstract of degree n* in L, and an *abstract corresponding to the formula* Φ. (A sentence is an abstract of degree 0).—By a *primitive abstract* we will mean an expression $\hat{x}_1 \ldots \hat{x}_n R_i^n (x_1, \ldots, x_n)$, where R_i^n is a relation symbol.

4.4. D. *Valuation of a class of relation-symbols:* If V is a class of relation-symbols, we mean (as for classes of terms in $\mathfrak{L}$)

[1] We use the symbols 'v_1', 'v_2', ..., 'u_1', 'u_2', ... as syntactic variables ranging over the class of individual variables x_1, x_2, ...

by a *valuation of V with universe U* a mapping M of V onto some class of relations with universe U, such that for every n the relation symbols of degree n are mapped on relations of degree n.

4.5. D. *Satisfaction, truth, model:* If L is a language in $PC_1=$, V the class of primitive symbols in L, M a valuation of V with universe U, and Φ a sentence in L, then M determines, in a familiar way, a truth-value of Φ, called the "truth-value of Φ under the interpretation M of the relation-symbols". And if Φ is any formula in L, with free individual variables $v_1, \ldots, v_n$, and $a_1, \ldots, a_n$ are objects in U, there is a familiar sense in saying that the assigment ϕ: $v_i \rightarrow a_i$ *satisfies* Φ in M. We do not here give a definition of this concept of satisfaction, but presuppose it as known. (See for instance Tarski (16)).

4.6. D. *Induced valuation, $T_{(A, M)}$, $T(M, A)$:* If L is a language, $V = V(L)$, M a valuation of V with universe U, and A an abstract $\hat{v}_1 \ldots \hat{v}_n \Phi$ in L, then by $T(M, A)$ ("the relation assigned by M to A"), we mean the relation R, with universe U, such that $R(a_1, \ldots, a_n) = 1$ iff the assignment ϕ: $v_i \rightarrow a_i$ satisfies Φ in M.—When A is of degree 0, $T(M, A)$ is the constant 1 or the constant 0, according to whether the truth-value of A determined by M is truth or falsity (see 4.5. D). If we let $\mathcal{A} = \mathcal{A}(L)$ be the class of all abstracts in L, (and M some valuation of $V(L)$), we see that the function $\lambda A.\ T(M, A)$ is a mapping of $\mathcal{A}$ onto a class of relations. We denote this mapping $T_{(A, M)}$, and call it "the valuation of $\mathcal{A}$ induced by M".

4.7. D. *Equivalence of abstracts:* We call two abstracts A_1, A_2 (in a language L) *equivalent*[1] abstracts, if for every valuation M of $V(L)$ it holds that $T(M, A_1) = T(M, A_2)$.

[1] Note that we use the term 'equivalent' in many different senses (see the definitions 4.13, 5.3, and 5.13), but that the sense intended is always clear from the context. Other terms, which are used in this systematically ambiguous way, are 'definable', 'embeddable' and 'regular'.

4.8. D. *The class* $\mathbf{Q}_0$ *of symbolic operations:* We will now introduce some symbolic operations which are applied to abstracts and give abstracts. They are closely related to the operations Q_0 on relations introduced in 2.6. D, and we use a notation which suggests the correspondence. The class of all the operations (1)–(6) listed below will be denoted $\mathbf{Q}_0$.

(1) (Place-permutation.) If A is an abstract $\hat{v}_1 \dots \hat{v}_n \Phi$, and σ is a permutation of the class $\{1, \dots, n\}$, we define $\mathbf{Per}_\sigma(A)$ to be the abstract $\hat{v}_{\sigma(1)} \dots \hat{v}_{\sigma(n)} \Phi$.

(2) (Negation.) If A is an abstract $\hat{v}_1 \dots \hat{v}_n \Phi$, we define $\mathbf{N}(A)$ to be the abstract $\hat{v} \dots \hat{v}_n (\sim \Phi)$.

(3) (Existential quantifier.) If A is an abstract $\hat{v}_1 \dots \hat{v}_n \Phi$, and $1 \leqslant k \leqslant n$, we define $\mathbf{E}_k(A)$ to be the abstract $\hat{u}_1 \dots \hat{u}_{n-1} ((\exists v_k) \Phi)$, where $u_i = v_i$ for $i < k$, and $u_i = v_{i+1}$ for $i \geqslant k$.

(4) (Identity.) If A is an abstract $\hat{v}_1 \dots \hat{v}_n \Phi$, and $r \neq s$, $1 \leqslant r, s \leqslant n$, we define $\mathbf{I}_{r,s}(A)$ to be the abstract $\hat{v}_1 \dots \hat{v}_n \cdot (\Phi \cdot v_r = v_s)$.

(5) (Merging.) If A is an abstract $\hat{v}_1 \dots \hat{v}_n \Phi$, and $r \neq s$, $1 \leqslant r, s \leqslant n$, we define $\mathbf{Id}_{r,s}(A)$ to be the abstract $\hat{u}_1 \dots \hat{u}_{n-1} \Phi'$, where $u_i = v_i$ for $i < s$, $u_i = v_{i+1}$ for $i \geqslant s$, and Φ' is obtained from Φ by substituting v_r for all free occurrences of v_s in Φ (and making changes of bound variables, if necessary).

(6) (Composition.) If A and B are abstracts, $A = \hat{v}_1 \dots \hat{v}_n \Phi(v_1, \dots, v_n)$, $B = \hat{u}_1 \dots \hat{u}_m \Psi(u_1, \dots, u_m)$, we define $\mathbf{Comp}(A, B)$ to be the abstract $\hat{x}_1 \dots \hat{x}_{n+m} (\Phi(x_1, \dots, x_n) \cdot \Psi(x_{n+1}, \dots, x_{n+m}))$.

4.9. T. If Φ and Ψ are formulas in $PC_1=$, and A_1 and A_2 are corresponding abstracts, then by applying successively one or several of the operations $\mathbf{Q}_0$ to A_1 and A_2, we may form abstracts equivalent to

(a) any abstract corresponding to the formula $\sim \Phi$.

(b) any abstract corresponding to the formula $(\exists x_i) \Phi$, where x_i is free in Φ.

(c) any abstract corresponding to the formula $(\Phi \cdot \Psi)$.

Proof: The proofs of (a) and (b) are almost immediate. One applies $\mathbf{N}$ and $\mathbf{E}_k$ (for suitable k), respectively, and pos-

sibly some place-permutation. For (c), one may first form **Comp** (A_1, A_2), then apply $\mathbf{Id}_{r,s}$ for suitable r, s, and then place-permutation.

Example: Let $A_1 = \hat{x}_1 \hat{x}_2 R_1(x_1 x_2)$, $A_2 = \hat{x}_2 \hat{x}_1 R_2(x_2 x_1)$, $B = \hat{x}_2 \hat{x}_1 (R_1(x_1 x_2) \cdot R_2(x_2 x_1))$. We will show how one may generate from A_1 and A_2 an abstract equivalent to B: We let $A_3 =$ **Comp** $(A_1, A_2) = \hat{x}_1 \hat{x}_2 \hat{x}_3 \hat{x}_4 (R_1(x_1 x_2) \cdot R_2(x_3 x_4))$; $A_4 = \mathbf{Id}_{1,4}(A_3) = \hat{x}_1 \hat{x}_2 \hat{x}_3 (R_1(x_1 x_2) \cdot R_2(x_3 x_1))$; $A_5 = \mathbf{Id}_{3,2}(A_4) = \hat{x}_1 \hat{x}_3 (R_1(x_1 x_3) \cdot R_2(x_3 x_1))$; and $A_6 = \mathbf{Per}_\sigma(A_5) = \hat{x}_3 \hat{x}_1 (R_1(x_1 x_3) \cdot R_2(x_3 x_1))$, where σ denotes the permutation $\binom{1\,2}{2\,1}$. The abstract A_6, which is thus generated from A_1 and A_2 by a succession of operations $\mathbf{Q}_0$, is clearly equivalent to B.

4.10. T: If A is any abstract of degree $\geqslant 1$, then by applying successively some of the operations $\mathbf{Q}_0$ to A we can form an abstract B equivalent to $\hat{x}_1 \hat{x}_2 (x_1 = x_2)$.

Proof: If A is an arbitrary abstract, we may form from A an abstract A' of degree 2 by either forming **Comp** (A, A), (in case A is of degree 1), or by applying $\mathbf{E}_1$ to A a number of times. From the abstract $A' = \hat{x}_i \hat{x}_j \Phi$ of degree 2 thus obtained we may form the abstract $\mathbf{I}_{1,2}\mathbf{N}\mathbf{I}\mathbf{d}_{1,3}\mathbf{I}\mathbf{d}_{2,4}$ **Comp** $(A', NA') = \hat{x}_1 \hat{x}_2 (\sim(\Phi(x_1 x_2) \cdot \sim \Phi(x_1 x_2)) \cdot x_1 = x_2)$, which is equivalent to $\hat{x}_1 \hat{x}_2 (x_1 = x_2)$.

Combining 4.9. T. and 4.10. T it is easy to prove the following theorem, which states that the set $\mathbf{Q}_0$ of operations is in a certain sense complete. The proof may be carried out by induction on the complexity of the abstract.

4.11. T: If L is any language in $PC_1=$ which contains at least one predicate symbol of degree $\geqslant 1$, and $A_1, A_2, \ldots$ are the primitive abstracts in L, then for any abstract A in L it is possible to form an equivalent abstract A' by a succession of applications of the operations $\mathbf{Q}_0$ to $A_1, A_2, \ldots$.

We will now define a sense of "equivalence", in which a term in $\mathfrak{L}$ and an abstract in $PC_1=$ can be said to be equivalent ("term-sense of equivalence"), and a related sense, in which an equation in $\mathfrak{L}$ and a sentence in $PC_1=$ can be said

to be equivalent ("sentence-sense of equivalence"). Our main result is the establishing of a kind of translatability between elementary languages, and languages in $PC_1=$.

We will first define what we mean by "corresponding" symbols, classes of primitive symbols, and valuations.

4.12. D. *"Corresponding" primitive symbols, classes of primitive symbols, valuations, valuation-classes, languages:* A variable r_m^n in $\mathfrak{L}$ and a relation symbol R_t^s are called *corresponding primitive symbols* iff $s=n$ and $t=m$. If V is a class of variables in $\mathfrak{L}$, and V' a class of relation symbols (in $PC_1=$), V and V' are called *corresponding classes of primitive symbols* if for every r_m^n, r_m^n is in V iff R_m^n is in V'. If V is a class of variables in $\mathfrak{L}$, and V' a class of relation symbols, M a valuation of V, and M' a valuation of V', then M and M' are called *corresponding valuations* if V and V' are corresponding symbol-classes, and, for every r_m^n in V, $M(r_m^n)=M'(R_m^n)$. If $\mathfrak{M}$ is a class of valuations of V and $\mathfrak{M}'$ a class of valuations of V', we call $\mathfrak{M}$ and $\mathfrak{M}'$ *corresponding valuation-classes* if $\mathfrak{M}$ contains the valuations corresponding to those in $\mathfrak{M}'$, and conversely. Two languages with corresponding sets of primitive symbols may be called *corresponding languages.*

4.13. D. *Equivalence between terms in $\mathfrak{L}$ and abstracts in $PC_1=$, and between equations in $\mathfrak{L}$ and sentences in $PC_1=$; corresponding systems:* Let t be any term in $\mathfrak{L}$, A any abstract in $PC_1=$. We say that t and A are *equivalent* (in the term-sense) if $T(M, t)=T(M', A)$ for any two corresponding valuations M and M' of the primitive symbols occurring in t and A. If E is an equation $t_1=t_2$ in $\mathfrak{L}$, and S a sentence in $PC_1=$, we say that E and S are *equivalent* (in the sentence-sense), if the truth-value of E determined by M (see 4.5. D) is the same as the truth-value of S determined by M', whenever M and M' are corresponding valuations of the primitive symbols involved. If $(L, \mathcal{E})$ is a system in $\mathfrak{L}$, and (L', Σ) is a system in $PC_1=$, we say that $(L, \mathcal{E})$ and (L', Σ) are *corresponding systems*, if L and L' are corresponding languages, and

the class $\mathcal{E}$ contains equations equivalent to every sentence in Σ, and Σ contains sentences equivalent to every equation in $\mathcal{E}$.

The main object of the present section is to prove the following two theorems.

4.14. T: Let L be any elementary language, and L' the corresponding language in $PC_1=$. Then to every term of L there exists an equivalent abstract in L', and conversely, to every abstract in L' there is an equivalent term in L.

4.15. T: Let L be any elementary language, and L' the corresponding language in $PC_1=$. Then to every equation of L there exists an equivalent sentence in L', and conversely, to every sentence in L' there exists an equivalent equation in L.

Remark. If we extended the definition of elementary language by removing the requirement (3.3. D) that it contain at least one variable of degree $\geqslant 1$, these theorems would not hold. On the other hand, because of this requirement, there is an elementary language corresponding to a language L in $PC_1=$ only if L contains at least one predicate symbol of degree $\geqslant 1$. We could obtain a more perfect correspondence between elementary languages and languages in $PC_1=$, by using a more complicated definition of elementary language. Even as the definitions stand, however, the correspondence is close enough to enable us to translate theorems about languages (and systems) in $PC_1=$ to theorems about elementary languages (and systems), and vice versa. We will see examples of this in section 6.

Proof of 4.14. T: We observe two facts: (a) If t is a primitive term r_m^n, then the primitive abstract $\hat{x}_1 \ldots \hat{x}_n R_m^n(x_1, \ldots, x_n)$ in L' is equivalent to t. (b) If the term t in $\mathcal{L}$ is equivalent to the abstract A, then $\boldsymbol{Per}_\sigma(t)$, $\boldsymbol{N}(t)$, $\boldsymbol{E}_k(t)$, $\boldsymbol{I}_{r,s}(t)$, $\boldsymbol{Id}_{r,s}(t)$ are equivalent to, respectively, $\mathbf{Per}_\sigma(A)$, $\mathbf{N}(A)$, $\mathbf{E}_k(A)$, $\mathbf{I}_{r,s}(A)$, and $\mathbf{Id}_{r,s}(A)$; and if t_1 is equivalent to A_1, and t_2 equivalent to A_2, then $\boldsymbol{Comp}(t_1, t_2)$ is equivalent to $\mathbf{Comp}(A_1, A_2)$.

From (a) and (b) it follows immediately by induction, that for any term t of L there is an equivalent abstract in L'. It follows also, that for any abstract A of L', which can be generated from the primitive abstracts by the operations $\mathbf{Q}_0$, there exists an equivalent term in L. But by a previous theorem (4.11. T) we know that for any abstract A in L there is an equivalent abstract A' which is so generated. (Since L' is a language corresponding to an elementary language, L' contains some relation symbol of degree $\geqslant 1$, and therefore theorem 4.11 is applicable. Cf. the preceding remark.) It follows that for any abstract A of L' there exists an equivalent term in L. This concludes the proof of 4.14. T.

Proof of 4.15. T: Let S be a sentence in L'. By the preceding theorem there exists a term t of degree 0 which is equivalent to S (in the "term-sense" of equivalence). But then the equation $t = \boldsymbol{NComp}\ (t, \boldsymbol{N}t)$ is an equation which is true for a valuation M iff $T(M, t) = 1$, i.e. this equation is equivalent to S (in the "sentence-sense" of equivalence).

On the other hand, let $t_1 = t_2$ be an equation in L (t_1 and t_2 terms of the same degree). By the preceding theorem there are abstracts A_1, A_2 in L' equivalent to t_1, t_2, respectively. We may choose these abstracts to contain identical prefixes, for instance $A_1 = \hat{x}_1 \ldots \hat{x}_n \Phi$, $A_2 = \hat{x}_1 \ldots \hat{x}_n \Psi$. The equation $t_1 = t_2$ is true for a valuation M iff $T(M', A_1) = T(M', A_2)$ for the corresponding valuation M'. However, the sentence $S = (x_1) \ldots (x_n)(\Phi \equiv \Psi)$ in L' is also true for M' iff $T(M', A_1) = T(M', A_2)$. (The expression $(x_1) \ldots (x_n)(\Phi \equiv \Psi)$ is here used as abbreviation for a more complicated formula in terms of negation, conjunction, and existential quantification.) We may conclude that S is a sentence equivalent to the original equation $t_1 = t_2$. This concludes the proof.

4.16. D. *The Boolean algebra $B(S, n)$ and the Boolean algebra $B(S', x_1, \ldots, x_n)$:* If $S = (L, \mathcal{E})$ is any system in $\mathfrak{L}$, we can define, for every n, a partial ordering relation among the n-degree terms of L by the condition: $t_1 \leqslant t_2$ iff for every model M of $(L, \mathcal{E})$, $M(t_1)$ is included in $M(t_2)$ ("inclusion"

among relations meant in the usual sense.) If $S=(L, \mathcal{E})$ is an elementary system, and $S'=(L', \sum)$ is a corresponding system in $PC_1=$, the analogous definitions evidently will give us essentially the same ordering of the n-degree abstracts in L'. However, we may observe that if $A=\hat{x}_1 \ldots \hat{x}_n \Phi$ and $B=\hat{x}_1 \ldots \hat{x}_n \Psi'$ are two n-degree abstracts, it will hold that $A \leqslant B$ according to this definition if and only if the sentence $(x_1) \ldots (x_n)(\Phi \supset \Psi')$ is a consequence of $\sum$. We see therefore that the partial ordering of abstracts in S' (and of terms in S) considered here is in effect identical with the partial ordering of $(x_1, \ldots, x_n)$-conditions by the relation of consequence, or derivability, in S'. We know that this is a Boolean algebra, and we denote it $B(S', x_1, \ldots, x_n)$.[1] The isomorphic algebra of terms defined by S we will designate $B(S, n)$.

[1] See (13), p. 92.

5. MODEL-THEORY FOR SYSTEMS IN $\mathfrak{L}$

The notion of model gives us an important relation between valuations on the one hand, and systems on the other hand. Model-theory may be described (vaguely) as the theory correlating formal ("syntactical") characteristics of systems, with structural characteristics of such valuations as correspond to these systems by the model-relation. (Also we might say that model-theory is the theory of the model-relation.)

Model-theoretic problems relating to elementary systems (in the form of $PC_1=$ -systems) have been intensively studied during the last ten years, and many significant results have been obtained. A classical and fundamental result in this field is Gödel's completeness theorem, which gives a formal criterion for an elementary system (system in $PC_1=$) to have a model (in our terminology: to by consistent). In section 6 we will cite some more results in the model theory of elementary systems. In the present section we will prove some model-theoretic results with regard to systems in the general sense (systems in $\mathfrak{L}$).

As a preliminary, we will first define a number of concepts relating only to valuations and valuation-classes.[1] Afterwards we will investigate corresponding syntactical notions.

5.1. D. *Isomorphic valuations; automorphisms of a valuation:* If C is any class of terms, and M any valuation of C with

[1] All the valuation-theoretic concepts introduced below in 5.1. D through 5.6. D, or at least concepts closely related to them, are discussed in more detail in (12). There are considerable terminological differences between the presentation in (12) and this presentation, (and it seems that the present terminology is much preferable). What is called a "relation class" (special case: relation) in (12) can be taken to be simply a variable-class (variable), where to each variable-class is associated a fixed regular class of valuations. Particular attention is paid to the relation D between relation classes (def. 25 in (12)); this corresponds to what we here call embeddability between valuation-classes (5.4. D).

universe U, and π is a one-one mapping of U onto a class U' then by $\pi(M)$ we will denote the valuation M' (with universe U'), defined by the condition $M'(t) = \pi M(t)$. The mapping π will be called an *isomorphism* between M and M', and M and M' are called *isomorphic valuations*. A permutation π of U is called an *automorphism* of M if $\pi(M) = M$; the class (group) of all such permutations is referred to as the *automorphism group* of M.

5.2. D. *Regular valuation-classes; regular mappings between valuation-classes:* Let $\mathfrak{M}$ be a class of valuations of some class C of terms. We will call $\mathfrak{M}$ a *regular valuation-class*, if whenever M and M' are isomorphic valuations of C and $M \in \mathfrak{M}$, then also $M' \in \mathfrak{M}$. Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be two regular classes of valuations. A mapping ϕ of the class $\mathfrak{M}_1$ into the class $\mathfrak{M}_2$ will be called a *regular mapping*, if $\phi(M)$ is always a valuation with the same universe as M, and if for any two valuations M, M' in $\mathfrak{M}_1$, such that $M' = \pi(M)$, $\phi(M') = \pi(\phi(M))$.

5.3. D. *Equivalence of valuation-classes:* Two valuation-classes $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are called *equivalent*, if there is a one-one regular mapping of $\mathfrak{M}_1$ onto $\mathfrak{M}_2$. (It is easy to see that the condition is symmetric.)

5.4. D. *Valuation-class embeddable in another valuation-class:* Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be two regular valuation-classes. If there is a subclass $\mathfrak{M}_1'$ of $\mathfrak{M}_1$, such that $\mathfrak{M}_1'$ and $\mathfrak{M}_2$ are equivalent, we will say that $\mathfrak{M}_2$ is *embeddable in* $\mathfrak{M}_1$.

5.5. D. *Valuation-class definable from another valuation-class:* If $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are two regular valuation-classes, we say that $\mathfrak{M}_2$ is *definable from* $\mathfrak{M}_1$, if there is a regular mapping of $\mathfrak{M}_1$ onto $\mathfrak{M}_2$.

5.6. D. *Restriction of valuation, restriction of valuation-class, $M[C]$, $\mathfrak{M}[C]$; compatible valuations, semantic definability:* If M is any valuation of C, and C' a subclass of C, we

use the notation $M[C']$ to denote the *restriction* of M to C', i.e. the valuation of C' which coincides with M on C'. If $\mathfrak{M}$ is a class of valuations of C, and C' a subclass of C, we mean by the *restriction* of $\mathfrak{M}$ to C' the class of restrictions to C' of members of $\mathfrak{M}$. We designate it $\mathfrak{M}[C']$. If $\mathfrak{M}$ is a class of valuations of C, and D_1, D_2 are two subclasses of C, two valuations M_1 and M_2 (of D_1 and D_2 respectively) are said to be *compatible in* $\mathfrak{M}$, if there is an M in $\mathfrak{M}$, such that $M_1 = M[D_1]$, and $M_2 = M[D_2]$. If for every M_1 in $\mathfrak{M}[D_1]$ there is only one valuation of D_2, which is compatible with M_1 in $\mathfrak{M}$, we will say that the term-set D_2 is *semantically defined* from the term-set D_1 in $\mathfrak{M}$. The mapping of $\mathfrak{M}[D_1]$ onto $\mathfrak{M}[D_2]$ determined by the relation of compatibility will be called the *definition mapping of* $\mathfrak{M}[D_1]$ *onto* $\mathfrak{M}[D_2]$ *determined by* $\mathfrak{M}$. It is easy to verify, that if $\mathfrak{M}$ is regular, then this definition mapping is a regular mapping. The most important case of semantic definability of a class D_2 from another class D_1 is where D_2 is a one-element class $\{t\}$. In this case we will say simply that *t is semantically defined from* D_1 *in* $\mathfrak{M}$.

The following theorem connects the two senses of "definable" defined above (5.5. D and 5.6. D). The proof of the theorem is easy, and is omitted.

5.7. T: When D_2 is semantically defined from D_1 in $\mathfrak{M}$, then $\mathfrak{M}[D_2]$ is definable from $\mathfrak{M}[D_1]$ in the sense of 5.5. D; and when $\mathfrak{M}_2$ (of D_2) is definable from $\mathfrak{M}_1$ (of D_1) in the sense of 5.5. D, and the classes D_1 and D_2 do not overlap, then there is a class $\mathfrak{M}$ of valuations of $D_1 \cup D_2$ such that $\mathfrak{M}_1 = \mathfrak{M}[D_1]$, $\mathfrak{M}_2 = \mathfrak{M}[D_2]$, and D_2 is semantically defined from D_1 in $\mathfrak{M}$.

5.8. D. *The model-class* $\mathfrak{M}(L, \mathcal{E})$ *and the equation-class* $\mathcal{E}(L, \mathfrak{M})$*; L-closure of a valuation-class, representable valuation-classes:* If $(L, \mathcal{E})$ is any system in $\mathfrak{L}$, we designate by $\mathfrak{M}(L, \mathcal{E})$ the class of all models of $(L, \mathcal{E})$; if L is a language, and $\mathfrak{M}$ is a class of valuations of $V(L)$, we designate by $\mathcal{E}(L, \mathfrak{M})$ the

class of all equations in L which hold for all valuations M in $\mathfrak{M}$. If $\mathfrak{M}$ is a valuation-class (of V), and $V(L) = V$, the class $\mathfrak{M}(L, \mathcal{E}(L, \mathfrak{M}))$ might be called the "*L-closure*" of $\mathfrak{M}$. It is a closure operation in the technical sense of the term. A valuation-class $\mathfrak{M}$ which can be represented as a class $\mathfrak{M}(L, \mathcal{E})$, (i.e. an "$L$-closed" valuation-class) will also be said to be *representable in L.*

5.9. T: Let V be a class of variables. Every class $\mathfrak{M}$ of valuations of V which is representable as $\mathfrak{M}(L, \mathcal{E})$, for some system $(L, \mathcal{E})$ in $\mathfrak{L}$, is regular.

Proof: Let M and $M' = \pi(M)$ be two isomorphic valuations of V. We will show that they make the same equations true in L. If E is an equation $t_1 = t_2$, E is true for a valuation M if $T(M, t_1) = T(M, t_2)$, and E is true for M', if $T(M', t_1) = T(M', t_2)$. But since all the operation symbols in L represent regular operations, we may infer, from the assumption that $M' = \pi(M)$, that for every t in L, $T(M', t) = \pi(T(M, t))$, and, since π is one-one, that $T(M, t) = \pi^{-1}(T(M', t))$. It follows that $T(M, t_1) = T(M, t_2)$ if and only if $T(M', t_1) = T(M', t_2)$, and therefore that M and M' make the same equations true.

5.10. T: (Converse of 5.9). Let V be a class of variables, and let $\mathfrak{M}$ be a regular class of valuations of V. Then there is some system $(L, \mathcal{E})$ in $\mathfrak{L}$, with $V(L) = V$, such that $\mathfrak{M}(L, \mathcal{E}) = \mathfrak{M}$.

Proof: (a) We deal first with the case that V is a finite class of variables $t_1, \ldots, t_n$. We define an n-ary operation P_1 (applicable to relation-sequences $(R_1, \ldots, R_n)$ with $d(R_1) = d(t_1), \ldots, d(R_n) = d(t_n)$), by the condition that $P_1(R_1, \ldots, R_n) = 1$ when there is an M in $\mathfrak{M}$ such that $M(t_i) = R_i$ for $i = 1, \ldots, n$, and $P_1(R_1, \ldots, R_n) = 0$ otherwise. (By 1 and 0 we mean here the constant relations of degree 0.) We also define an operation P_2 such that $P_2(R_1, \ldots, R_n) = 1$ for all values of $R_1, \ldots, R_n$ (with $d(R_1) = d(t_1), \ldots, d(R_n) = d(t_n)$). If L is any language containing the operation-symbols p_1 and p_2 corresponding to these (regular) operations, and E is the equation $p_1(t_1, \ldots, t_n) =$

$p_2(t_1, \ldots, t_n)$, then the system $S = (L, \{E\})$ has the desired property that $\mathfrak{M}(S) = \mathfrak{M}$.

(b) We let now V be an infinite class $\{t_1, t_2, \ldots\}$ of variables. We denote for every n the subclass $\{t_1, \ldots, t_n\}$ by V_n, and we let $\mathfrak{M}_n$ be the class of all valuations in $\mathfrak{M}$ restricted to V_n. According to part (a) of the proof, there are systems $S_n = (L_n, \mathcal{E}_n)$ such that $\mathfrak{M}(S_n) = \mathfrak{M}_n$. If we let L be the sum (cf. 3.3. D) of all L_n, and $\mathcal{E}$ the union of all $\mathcal{E}_n$, the resulting system $S = (L, \mathcal{E})$ will have the desired property that $\mathfrak{M}(S) = \mathfrak{M}$.

The following theorem relates the semantic sense of definability (5.6. D), and the syntactic sense of definability (3.9. D).

5.11. T: Let $S = (L, \mathcal{E})$ be a system, and let C be a finite class of terms in L, and t_0 a term in L, such that t_0 is semantically defined in $\mathfrak{M}(L, \mathcal{E})$ from C. Then it is always possible to find an extension $S' = (L', \mathcal{E})$ of the system S, where L' is formed from L by adding an operation symbol, such that t_0 is syntactically defined from C in S', (and such that $\mathfrak{M}(S') = \mathfrak{M}(S)$).

Proof: Let the terms in C be $t_1, \ldots, t_k$. We define a k-ary mapping P (on relations) as follows: For any model M of S with universe U, when $M(t_0) = R_0$, $M(t_1) = R_1, \ldots, M(t_k) = R_k$, we let $P(R_1, \ldots, R_k) = R_0$. (According to the assumption of semantic definability, this condition gives at most one R_0 corresponding to one k-tuple $(R_1, \ldots, R_k)$). For any other sequence $(R_1, \ldots, R_k)$ with $d(R_i) = d(t_i)$ for $i = 1, \ldots, k$, we let $P(R_1, \ldots, R_k)$ be the universal relation of degree $d = d(t_0)$, i.e. the relation R with universe U such that $R(a_1, \ldots, a_d) = 1$ for all $a_1, \ldots, a_d$ in U. The mapping thus defined can be seen to be regular. We now form the language L' by adding to L the operation symbol p corresponding to the operation P. We have constructed P in such a way that the equation $t_0 = p(t_1, \ldots, t_k)$ must be a consequence of $\mathcal{E}$. The condition $\mathfrak{M}(S) = \mathfrak{M}(S')$ is fulfilled because the variables in S and S' are the same, and the equations (axioms) are the same. This concludes the proof.

5.12. T: It is very easy to prove, as a converse of 5.11, that if t is syntactically defined in terms of C in S, then t is also semantically defined from C in $\mathfrak{M}(S)$. We may sum up these two results by saying that one may equate (as regards finite term-classes) the notions "semantically defined", and "syntactically defined in a sufficiently strong $\mathfrak{L}$-language".

We will now introduce syntactical concepts corresponding to the valuation-theoretic concepts of 5.4–5.6, and prove some theorems about the correspondence.

5.13. D. *Equivalence of systems:* Let $(L, \mathcal{E})$ and $(L', \mathcal{E}')$ be two systems in $\mathfrak{L}$, such that L and L' have the same operation symbols. We will call these systems *equivalent systems*, if there is a relation R ("equivalence relation"), with the following properties:

(a) The domain of R is L, the converse domain is L'.

(b) If $R(t, t')$, then t and t' are terms of the same degree.

(c) If $R(t_1, t_1')$, and $t_2 \in L$, $t_2' \in L'$ then $R(t_2, t_1')$ iff $t_1 = t_2$ is a consequence of $\mathcal{E}$, and $R(t_1, t_2')$ iff $t_1' = t_2'$ is a consequence of $\mathcal{E}'$.

(d) If $t_1, \ldots, t_k$ and $p(t_1, \ldots, t_k)$ are terms in L, and $R(t_i, t_i')$ for $i = 1, \ldots, k$, then $R(p(t_1, \ldots, t_k), p(t_1', \ldots, t_k'))$.

Explanation: We have equivalent systems in this sense in $PC_1=$, when we have two systems which may be regarded as different formulations (with different choices of primitives) of the same theory; for instance, one system for linear order in terms of $\leqslant$, and another system in terms of $<$. The equivalence relation R can then be taken as the ordinary "translation" of abstracts between the two systems.

5.14. D. *Strengthening, subsystem; system definable from, embeddable in, another system:* (a) A system $(L', \mathcal{E}')$ is called a *strengthening* of another system $(L, \mathcal{E})$, if $L = L'$ and $\mathcal{E} \subseteq \mathcal{E}'$. (b) A system $(L', \mathcal{E}')$ is called a *subsystem* of another system $(L, \mathcal{E})$, if $Q(L') = Q(L)$, $V(L') \subseteq V(L)$, and if an equation in L' is a consequence of $\mathcal{E}'$ if and only if it is a consequence of $\mathcal{E}$. (c) A system S' is said to be *embeddable* in another system S, if

S' is equivalent to some strengthening of S. (d) A system S' said to be *definable from* S, if S' is a subsystem of some system equivalent to S.

5.15. T: If $(L, \mathcal{E})$ and $(L' \mathcal{E}')$ are two equivalent systems, and $\mathfrak{M}$ and $\mathfrak{M}'$ are the corresponding model-classes (i.e. $\mathfrak{M} = \mathfrak{M}(L, \mathcal{E})$ and $\mathfrak{M}' = \mathfrak{M}(L', \mathcal{E}')$) then $\mathfrak{M}$ and $\mathfrak{M}'$ are equivalent in the sense of definition 5.3, i.e. there is a one-one regular mapping of $\mathfrak{M}$ onto $\mathfrak{M}'$.

Proof: If we let R be an equivalence relation between $(L, \mathcal{E})$ and $(L', \mathcal{E}')$, we define a relation ϕ between valuations in $\mathfrak{M}$ and in $\mathfrak{M}'$ as follows: $\phi(M, M')$ iff $T(M, t) = T(M', t')$ for all $t \in L$, $t' \in L'$ such that $R(t, t')$. It is not difficult to verify that this relation gives a one-one regular mapping between $\mathfrak{M}$ and $\mathfrak{M}'$.

A restricted converse of this theorem is the following:

5.16. T: Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be two equivalent regular valuation-classes, of the variable-sets V_1 and V_2 respectively, where V_1 and V_2 are finite. Then there are equivalent systems $(L_1, \mathcal{E}_1)$ and $(L_2, \mathcal{E}_2)$, with $V(L_1) = V_1$ and $V(L_2) = V_2$, such that $\mathfrak{M}_1 = \mathfrak{M}(L_1, \mathcal{E}_1)$ and $\mathfrak{M}_2 = \mathfrak{M}(L_2, \mathcal{E}_2)$.

Proof: We will assume that V_1 and V_2 have no common elements. (This assumption will have no essential effect on the strength of the theorem.) We let V be the union $V_1 \cup V_2$, and we let ϕ be some given regular one-one mapping of $\mathfrak{M}_1$ onto $\mathfrak{M}_2$. We now define a class $\mathfrak{M}$ of valuations of V as follows: If M is a valuation of V, then $M \in \mathfrak{M}$ iff there is an M_1 in $\mathfrak{M}_1$ and an M_2 in $\mathfrak{M}_2$ such that $M[V_1] = M_1$, (for the restriction of M to V_1—see 5.6. D), $M[V_2] = M_2$, and $\phi(M_1, M_2)$. This valuation-class $\mathfrak{M}$ is regular. We can also see that in $\mathfrak{M}$ each t in V_1 is semantically defined from V_2, and each t in V_2 is semantically defined from V_1. Therefore, by 5.10. T and 5.11. T, we can find a system $S = (L, \mathcal{E})$ with V as its class of variables, such that $\mathfrak{M}(S) = \mathfrak{M}$, and such that each t in V_1 is syntactically defined from V_2 in S, and each

t in V_2 is syntactically defined from V_1 in S. This implies that each of the classes V_1 and V_2 is a generating set of the system $(L, \mathcal{E})$ (3.9. D). We define the system S_1 to be $(L_1, \mathcal{E}_1)$, and S_2 to be $(L_2, \mathcal{E}_2)$, where L_1 is the part of L restricted to the variables V_1, L_2 the part of L restricted to the variables V_2, and where $\mathcal{E}_1$ is the class of consequences of $\mathcal{E}$ in L_1 and $\mathcal{E}_2$ the class of consequences of $\mathcal{E}$ in L_2. The relation R between terms in L_1 and terms in L_2 is now defined by the condition: $R(t, t')$ iff the equation $t = t'$ is a consequence of $\mathcal{E}$. This relation R is an equivalence relation between the systems S_1 and S_2 in the sense of 5.13. D. Also it can easily be verified that $\mathfrak{M}(S_1) = \mathfrak{M}_1$, and $\mathfrak{M}(S_2) = \mathfrak{M}_2$. This concludes the proof.

5.17. T: Let S and S' be two systems, such that S' is definable from S (5.14. D). Then if $\mathfrak{M}$ and $\mathfrak{M}'$ are the corresponding model-classes, $\mathfrak{M}'$ is definable from $\mathfrak{M}$ (i.e. there exists a regular mapping of $\mathfrak{M}$ onto $\mathfrak{M}'$).

Proof: This follows from 5.15 T together with the observation that there is a natural regular mapping in the case when S' is a subsystem of S.

5.18. T: (Restricted converse of 5.17). Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be regular valuation-classes of V_1 and V_2, respectively, where V_1 is a finite, V_2 an arbitrary variable-class, and let $\mathfrak{M}_2$ be definable from $\mathfrak{M}_1$. Then there are systems S_1 and S_2 such that $\mathfrak{M}(S_1) = \mathfrak{M}_1$, and $\mathfrak{M}(S_2) = \mathfrak{M}_2$, and such that the system S_2 is definable from S_1.

Proof: The proof is a modification of that for theorem 5.16.

If S and S' are two systems such that S' is a strengthening of S, and $\mathfrak{M}$ and $\mathfrak{M}'$ are corresponding model-classes, then $\mathfrak{M}'$ is a subclass of $\mathfrak{M}$. The following theorem is a converse of this.

5.19. T: Let V be a class of variables, let $\mathfrak{M}$ be a regular set of valuations of V, and let $\mathfrak{M}'$ be a regular subset of $\mathfrak{M}$.

Then there is some system S, and a strengthening S' of S, such that $\mathfrak{M}(S) = \mathfrak{M}$ and $\mathfrak{M}(S') = \mathfrak{M}'$.

Proof: By 5.10. T, there are systems $S_1 = (L_1, \mathcal{E}_1)$ and $S_2 = (L_2, \mathcal{E}_2)$ with $V(L_1) = V(L_2) = V$, such that $\mathfrak{M}(S_1) = \mathfrak{M}_1$ and $\mathfrak{M}(S_2) = \mathfrak{M}_2$. We let L be the sum of the languages L_1 and L_2 (i.e.: $V(L) = V$, $Q(L) = Q(L_1) \cup Q(L_2)$), we let $\mathcal{E} = \mathcal{E}_1$ and we let $\mathcal{E}'$ be the union $\mathcal{E}_1 \cup \mathcal{E}_2$. We let $S = (L, \mathcal{E})$, and $S' = (L, \mathcal{E}')$. The systems S and S' so defined satisfy the requirements of the theorem.

We now show the correspondence between the syntactical and the semantical senses of embeddability.

5.20. T: Let S_1 and S_2 be two systems such that S_2 is embeddable in S_1. Then if $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are the corresponding model-classes, $\mathfrak{M}_2$ is embeddable in $\mathfrak{M}_1$ (i.e. there is a regular one-one mapping of $\mathfrak{M}_2$ onto some subclass of $\mathfrak{M}_1$.)

Proof: By definition, there is a strengthening S_1' of S_1 such that S_1' is equivalent to S_2. Then, according to 5.15. T, if $\mathfrak{M}_1'$ is the model-class of S_1', there is a one-one mapping of $\mathfrak{M}_2$ onto $\mathfrak{M}_1'$. But $\mathfrak{M}_1'$ is a subclass of $\mathfrak{M}_1$. This proves the theorem.

5.21. T: (Restricted converse of 5.20.) Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be regular valuation-classes, of V_1 and V_2 respectively, where V_1 and V_2 are finite variable-classes, such that $\mathfrak{M}_2$ is embeddable in $\mathfrak{M}_1$. Then there are systems $S_1 = (L_1, \mathcal{E}_1)$, and $S_2 = (L_2, \mathcal{E}_2)$, with $V(L_1) = V_1$ and $V(L_2) = V_2$, $\mathfrak{M}(S_1) = \mathfrak{M}_1$, $\mathfrak{M}(S_2) = \mathfrak{M}_2$, such that S_2 is embeddable in S_1.

Proof: There is a regular subclass $\mathfrak{M}_3$ of $\mathfrak{M}_1$ which is equivalent to $\mathfrak{M}_2$. According to 5.19. T, there are systems $(L_1, \mathcal{E}_1)$ and $(L_1, \mathcal{E}_3)$ corresponding to $\mathfrak{M}_1$ and $\mathfrak{M}_3$, such that $(L_1, \mathcal{E}_3)$ is a strengthening of $(L_1, \mathcal{E}_1)$. According to 5.16. T, there are equivalent systems $(L_2, \mathcal{E}_2)$ and $(L_3, \mathcal{E}_3')$ corresponding to $\mathfrak{M}_2$ and $\mathfrak{M}_3$. If we let L_1' and L_2' be the extensions of the languages L_1 and L_2, obtained by letting $Q(L_1') = Q(L_2') = Q(L_1) \cup$

$Q(L_2)$, the systems $S_1 = (L_1', \mathcal{E}_1)$ and $S_2 = (L_2', \mathcal{E}_2)$ will satisfy the requirements of the theorem.

In (12), we state and prove a number of theorems about the valuation-theoretic relation of embeddability (there denoted "D", see def. 25 in (12)). As an illustration of the correspondence between semantical and syntactical concepts just demonstrated, we will here restate one of these theorems in syntactic form.

5.22. T: Let L_1 and L_2 be two languages in $\mathfrak{L}$ such that, for some n and k, $V(L_1) = \{r_1^{n+1}\}$, $V(L_2) = \{r_1^n, \ldots, r_k^n\}$. Then the system $S_1 = (L_1, \Lambda)$ is not embeddable in $S_2 = (L_2, \Lambda)$. ('Λ' represents the empty class of equations.)

Proof: According to 5.20. T, if S_1 were embeddable in S_2, $\mathfrak{M}(S_1)$ would be embeddable in $\mathfrak{M}(S_2)$. But $\mathfrak{M}(S_1)$ is the class of all valuations of $\{r_1^{n+1}\}$, and $\mathfrak{M}(S_2)$ is the class of all valuations of $\{r_1^n, \ldots, r_k^n\}$. Therefore, theorem 22 in (12) can be taken to state that $\mathfrak{M}(S_1)$ is not embeddable in $\mathfrak{M}(S_2)$. We have then shown that the system S_1 is not embeddable in the system S_2. This concludes the proof of the theorem.

6. MODEL-THEORY FOR ELEMENTARY SYSTEMS

All the theorems cited below are originally stated for systems in $PC_1=$. However, by using the theorems in section 4, it is easy to make the transition to a formulation in terms of elementary systems.

We know that for an arbitrary system S, if a term t is syntactically defined in S from a class C (finite), then it is also semantically defined from C in $\mathfrak{M}(S)$. (See 5.12). For elementary systems the converse also holds. This result (first proved by Beth[1]) may be formulated as follows:

6.1. T: Let $(L, \mathcal{E})$ be an elementary system, and let t_0, $t_1, \ldots, t_k$ be terms in L such that t_0 is not syntactically defined from $t_1, \ldots, t_k$ in $(L, \mathcal{E})$. Then there are models M_1, M_2 of $(L, \mathcal{E})$ such that $M_1(t_i) = M_2(t_i)$ for $i = 1, \ldots, k$ and $M_1(t_0) \neq M_2(t_0)$.

For complete systems, we may strengthen this result in the following way. (A proof of this result, in a slightly different formulation, may be found in (14).)

6.2. T: Let $(L, \mathcal{E})$ be a complete elementary system, and let $t_0, t_1, \ldots, t_k$ be terms in L such that t_0 is not syntactically defined from $t_1, \ldots, t_k$ in $(L, \mathcal{E})$. Then there is some model M of $(L, \mathcal{E})$ with universe U, and a permutation π of U, such that $\pi M(t_i) = M(t_i)$ for $i = 1, \ldots, k$, and $\pi M(t_0) \neq M(t_0)$. (In other words, we may choose two isomorphic models as the models M_1 and M_2 of the preceding theorem.)

A problem which suggests itself is whether the analogues of 6.1. T hold for the notions of "equivalence" and "embeddability": When S and S' are elementary systems, and $\mathfrak{M}$ and $\mathfrak{M}'$ are the corresponding model-classes, is it always true

[1] See (1), (2), (8), and (10).

that S is equivalent to (embeddable in) S' when $\mathfrak{M}$ is equivalent to (embeddable in) $\mathfrak{M}'$? We leave this question unanswered.

6.3. D. *Elementarily indistinguishable valuations; complete elementary valuation-classes:* In the general theory of $\mathfrak{L}$-systems we saw that a valuation-class is always representable if it is regular. When we restrict ourselves to elementary languages, this is, as would be expected, no longer the case. If M_1, M_2 are two non-isomorphic valuations of V, it may even happen that the same equations in $L_0(V)$ are true for M_1 and for M_2. In this case we will call M_1 and M_2 *elementarily indistinguishable* valuations. A valuation-class $\mathfrak{M}$ consisting of those valuations which are elementarily indistinguishable from some one valuation may be called a "*complete elementary class*". If $\mathfrak{M} = \mathfrak{M}(S)$, where S is an elementary system, we can see that $\mathfrak{M}$ is complete if and only if S is a complete system. We are interested in the structure of these "complete" valuation-classes.

6.4. D. *Minimal model of a system:* We know that if $(L, \mathcal{E})$ is an elementary system, and M a model of $(L, \mathcal{E})$ then every L-enlargement M' of M is also a model, and every valuation isomorphic to such an enlargement is again a model. If we can obtain all the models of $(L, \mathcal{E})$ in this manner, we may say that $\mathfrak{M}(L, \mathcal{E})$ is "generated" from M by the relations of L-extension and isomorphism, and we will call M a *minimal model* of $(L, \mathcal{E})$. In (15), besides some other theorems on minimal models, we prove the following theorem, which states the conditions under which a complete elementary system has a model-set of this particularly simple type.

6.5. T:[1] Let S be a complete elementary system. Then there is a model M of S (minimal model), such that every other model of S is isomorphic to some L-enlargement of M, if and only if, for every n, the Boolean algebra $\mathrm{B}(S, n)$ is atomic.

[1] It has been pointed out to me (by Dr. Craig), that this result is also announced by R. L. Vaught in *Notices of the American Math. Soc., Nov. 1958.*

6.6. D. *Categorical systems:* If a consistent system does not have any non-isomorphic models, we call it a *categorical system.* It can be shown that an elementary system can be categorical only if it is the description of some finite valuation (valuation with finite universe). We may say that this is due to the impossibility to distinguish, in an elementary language, between different infinite powers (cardinalities). Or more exactly:

6.7. T: If an elementary system S has a model of infinite power c_1, and c_2 is any other infinite power, then S has also some model of power c_2.[1] (See for instance Łoś (7, T. 9)).

A weaker concept than "categorical" is "categorical in power c", or "c-categorical", which is defined as follows:

6.8. D. *c-categorical systems:* A system S is said to be *c-categorical* (where c is any cardinal number), if any two models M_1 and M_2 of S of power c are isomorphic.

Of special interest is the case $c = \aleph_0$. In (13) there is a proof of a theorem first proved by Ryll-Nardzewski[2] which states a necessary and sufficient syntactic condition for $\aleph_0$-categoricity. We will list below (6.11. T) one of the versions of this theorem stated in (13), and a new slightly different version (6.12. T), which gives a condition for $\aleph_0$-categoricity of a system in terms of the automorphism group of the model described by the system. Before giving the theorems, we list some definitions.

6.9. D. *L-isomorphic sequences in a valuation; L-types:* When M is a valuation of V with universe U, and L a language with $V(L) = V$, two sequences $s_1 = (a_1, \ldots, a_n)$ and $s_2 = (b_1, \ldots, b_n)$ of elements in U are said to be *L-isomorphic* in M, if for every term t in L, s_1 satisfies t in M if and only if s_2 satisfies t in M. By an *L-type* of n-sequences in M we mean a class consisting of all n-sequences in U^n which are L-isomorphic to some one common sequence.

---

[1] By the *power* of a valuation or model M we mean the power of the universe of M.—A *denumerable valuation* is a valuation of power $\aleph_0$.

[2] See (9) and (11).

6.10. D. *Permutable sequences in a valuation; structural types:* When M is a valuation of V with universe U, and $a_1, \ldots, a_n$, $b_1, \ldots, b_n$ are elements in U, we say that the sequences $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ are *permutable with each other*, if there is an automorphism π of M such that $\pi(a_1) = b_1, \ldots, \pi(a_n) = b_n$. By a *structural type* of n-sequences in M we mean a class consisting of all the n-sequences in U^n which are permutable with some one common n-sequence.

6.11. T: Let M be a denumerable valuation of V, let $L = L_0(V)$ (i.e. the elementary language with variable-set V), and let $\mathcal{E} = \mathcal{E}(L, \{M\})$. Then the (complete) system $(L, \mathcal{E})$ is $\aleph_0$-categorical iff for each n the number of different L-types of n-sequences in M is finite.

(As mentioned above, this theorem is a reformulation of theorem 11 in (13).)

6.12. T: Let M be a denumerable valuation of V, let $L = L_0(V)$, and let $\mathcal{E} = \mathcal{E}(L, \{M\})$. Then the system $(L, \mathcal{E})$ is $\aleph_0$-categorical, iff for each n the number of different structural types of n-sequences in M is finite.

Proof of 6.12. T: The if-part follows almost immediately from the preceding theorem. If two sequences in M belong to the same structural type, then they clearly also belong to the same L-type. Therefore, if for each n the number of structural types of n-sequences is finite, the number of L-types is also finite. And then, according to the preceding theorem, the system is $\aleph_0$-categorical. To prove the implication the other way, we assume that the system is $\aleph_0$-categorical. Then we know from the preceding theorem that for given n, the number of L-types of n-sequences is finite. We want to show that the number of structural types of n-sequences is also finite. It is enough to show that if two n-sequences have the same L-type in M, they belong to the same structural type. This follows easily from a number of theorems in (13), as follows: Since the system $(L, \mathcal{E})$ is $\aleph_0$-categorical, M is of

finite character (see (13), T 9 and def. of "finite character" p. 85); therefore (see (13), parts (b) and (c) of the proof of T 3) any partial isomorphism of M into itself can be extended to a complete isomorphism (automorphism). This is, however, only another way of saying that two sequences of the same L-type in M belong to the same structural type. This concludes the proof of the theorem.

LIST OF DEFINITIONS AND NOTATIONS

(in order of appearance)

2.1. *Relation of degree n with universe U; d (R), U (R).*
2.2. *Isomorphic relations.*
2.3. *Automorphism; automorphism group.*
2.4. *Regular operations on relations.*
2.5. *Type-sequence of a (regular) operation.*
2.6. *The class Q_0 of fundamental elementary operations.*
3.1. *Variables, operation-symbols, terms in $\mathfrak{L}$; d (t).*
3.2. *The class Q_0 of elementary operation-symbols; elementary terms.*
3.3. *Generated terms, Gen (C, Q); language in $\mathfrak{L}$, sum of languages, elementary languages; V (L), Q (L), L_0 (V).*
3.4. *Equation; equation in L.*
3.5. *Valuation, consistent valuation.*
3.7. *Induced valuation, $T_{(C,M)}$, T (M, t).*
3.8. *"Hold for", "true for", defined for equations and sets of equations; consequence.*
3.9. *System in $\mathfrak{L}$; term syntactically defined from term-set; generating set of terms in a system.*
3.10. *Model of a system.*
3.11. *Consistent system, complete system.*
3.12. *Sequences which satisfy a term.*
3.13. *Enlargement of a valuation.*
3.14. *L-enlargement.*
4.1. *Individual variables, relation symbols, formulas, sentences* in $PC_1=$.
4.2. *Language in $PC_1=$, primitive symbol; system in $PC_1=$.*
4.3. *Abstract in L; primitive abstract.*
4.4. *Valuation of a class of relation-symbols.*
4.5. *Satisfaction, truth, model.*
4.6. *Induced valuation, $T_{(A,M)}$, T (M, A).*
4.7. *Equivalence of abstracts.*
4.8. *The class Q_0 of symbolic operations.*
4.12. *"Corresponding" primitive symbols, classes of primitive symbols, valuations, valuation-classes, languages.*
4.13. *Equivalence between terms in $\mathfrak{L}$ and abstracts in $PC_1=$; equivalence between equations in $\mathfrak{L}$ and sentences in $PC_1=$; corresponding systems.*

4.16. *The Boolean algebra* $B(S, n)$ *and the Boolean algebra* $B(S', \mathfrak{x}_1, \ldots, \mathfrak{x}_n)$.
5.1. *Isomorphic valuations, automorphisms of a valuation.*
5.2. *Regular valuation-classes; regular mappings between valuation-classes.*
5.3. *Equivalence of valuation-classes.*
5.4. *Valuation-class embeddable in another valuation-class.*
5.5. *Valuation-class definable from another valuation-class.*
5.6. *Restriction of valuation, restriction of valuation-class,* $M[C]$, $\mathfrak{M}[C]$; *compatible valuations, semantic definability.*
5.8. *The model-class* $\mathfrak{M}(L, \mathcal{E})$ *and the equation-class* $\mathcal{E}(L, \mathfrak{M})$; *L-closure of a valuation-class, representable valuation-classes.*
5.13. *Equivalence of systems.*
5.14. *Strengthening, subsystem; system definable from, embeddable in, another system.*
6.3. *Elementarily indistinguishable valuations; complete elementary valuation-classes.*
6.4. *Minimal models of a system.*
6.6. *Categorical systems.*
6.8. *c-categorical systems.*
6.9. *L-isomorphic sequences in a valuation: L-type.*
6.10. *Permutable sequences in a valuation; structural types.*

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